Accommodation of Kinematic Disturbances During Minimum-Time Maneuvers of Flexible Spacecraft

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This paper is concerned with control of the perturbations experienced by a flexible spacecraft during a minimum-time maneuver. The spacecraft is modeled as a rigid hub with a flexible appendage. The perturbations consist of deviations from a rigid-body maneuver and elastic vibration. The vibration is described by a linear, time-varying set of ordinary differential equations subjected to piecewise-constant disturbances caused by inertial forces resulting from the maneuver. The control is carried out during the maneuver period, which is relatively short, and it uses an observer to estimate the controlled state and part of the disturbance vector. The controller is divided into an optimal finite-time linear quadratic regulator for the reduced-order model and a disturbance-accommodation control that minimizes a weighted norm spanning the constant part of the full-modeled state. The controller is designed to mitigate the effects of control and observation spillover, as well as of modeling errors. The developments are illustrated by means of a numerical example.

I. Introduction

THIS paper is concerned with the vibration control during a minimum-time slewing maneuver of a flexible spacecraft consisting of a rigid hub and a flexible appendage, where the appendage is in the form of a beam with one end attached to the hub and the other end free (Fig. 1). The equations of motion represent a hybrid set of nonlinear ordinary differential equations and partial differential equations.[1] Practical reasons dictate discretization in space and truncation, which is ordinarily carried out by representing the elastic motions as finite linear combinations of space-dependent admissible functions multiplied by time-dependent generalized coordinates.[2] The net result is a set of nonlinear ordinary differential equations of high order.

In recent years, there has been considerable interest in control of maneuvering space structures. Single-axis slewing maneuver about a principal axis of a space structure consisting of a rigid hub and a number of flexible appendages has been investigated in Refs. 3–5 by means of a reduced-order nonlinear model. The solution, satisfying prescribed final conditions, minimizes a performance measure in the form of a combination of control effort and energy, where the final time is fixed but otherwise arbitrary. The solution of the nonlinear two-point boundary-value problem (TPBVP) encountered in the minimum-time problem of the nonlinear reduced-order model is approximated by means of the continuation method and yields a closed-loop solution for the reduced-order model. In Ref. 6, the control is restricted in that only three switching times can be selected, where the final time is prescribed a priori. These three switching points are determined so as to minimize the postmaneuver elastic energy, whereas the minimization problem is constrained by prescribed angular impulse and its time integral. The solution results in on-off, open-loop control. The problem in which the cost is a combination of fuel and transition time was investigated in Ref. 7 by means of a nonmaneuvering reduced-order linear model, and expressions for the switching points are obtained by an approximate method. This also is on-off, open-loop control.

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Near time-optimal control minimizing the time and a certain measure of residual energy was investigated in Ref. 8. The control is again open loop. The slewing problem in which the cost is the transition time only was investigated in Refs. 9–11. In these investigations, a time-optimal problem is solved for a nonmaneuvering reduced-order model, resulting in a linear model. In Ref. 11, this solution serves as an initial guess for the solution of the nonlinear TPBVP describing the maneuvering of the reduced-order model.

Design of minimum-time control for the system just mentioned involves the solution of a nonlinear two-point boundary-value problem of a very high order, which is not feasible for on-line computations. However, because the nonlinearity enters through the rigid-body motions, while the elastic motions tend to be small, the difficulties can be circumvented by adopting a perturbation approach. This approach yields a zero-order nonlinear model of low order describing the "rigid-body" maneuver of the spacecraft and a first-order linear time-varying model subjected to persistent disturbances consisting of deviations from the rigid-body slewing and elastic vibrations.[12,13] The zero-order control is carried out in minimum time, which implies bang-bang control. On the other hand, the first-order control, which is the main concern of this paper, represents linear feedback control.

The problem of vibration control during maneuver has been considered before in the context of the perturbation approach.[13,14] In Ref. 13, the maneuver period is divided into several time intervals, and a time-invariant control law based on the disturbance-free model is applied in every interval. As a result, the bulk of the vibration suppression takes place after the termination of the maneuver. In Ref. 14, an optimal control law incorporating integral control is designed on the basis of the time-varying perturbed model. Perfect observation is assumed, so that only control spillover affects the response of the full model.

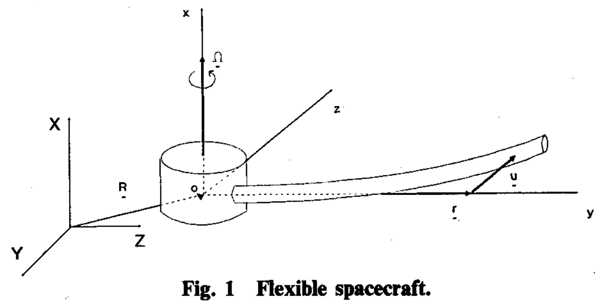

Fig. 1 Flexible spacecraft.

In this paper, the control policy is based on a reduced-order compensator, which consists of a Luenberger observer and a controller. The observer estimates part of the modeled state, which includes the controlled states and the observed but uncontrolled states, as well as part of the disturbance vector. The controller is divided into two parts, as suggested in Refs. 15 and 16, where the first part is designed to enhance the dynamic characteristics of a reduced-order model and the second part accommodates the persistent disturbances.

The control problem can be divided into three parts:

1) A reduced-order model is to be stabilized according to a quadratic performance measure within the finite-time interval of the maneuver.

2) A weighted norm of the constant part of the disturbance response of the first-order model is to be minimized in quasi-time-invariant conditions imposed on the first-order model toward the end of the finite-time interval.

3) The supremum time constant of the first-order model is to be minimized without compromising the above quasi-time-invariant conditions.

From the preceding we conclude that the feedback control is designed to stabilize the reduced-order model, to impose quasi-time-invariant conditions on the first-order model, and, at the same time, to minimize the supremum-time constant of the first-order model.

In this paper, we concentrate mainly on the first two aspects of the problem, whereas the third one is summarized only briefly. A detailed analysis of the third aspect can be found in Refs. 17 and 18.

II. Equations of Motion

The perturbation equations are described in Ref. 3 and can be simplified by introducing a linear transformation involving the eigensolution of the nonmaneuvering spacecraft. These equations are referred to as quasimodal equations and have the form[14]

$$\ddot{v} + (2\Omega_0\bar{G} + \bar{D})\dot{v} + (\Lambda + \dot{\Omega}_0\bar{G} - \Omega_0^2\bar{K})v = \bar{E}T - \dot{\Omega}_0\Psi \quad \text{(1a)}$$

$$y_d = \bar{C}v, \qquad y_v = \bar{C}\dot{v} \quad \text{(1b)}$$

where $v = [v_t^T \,|\, v_r^T \,|\, v_e^T]^T$ is the vector of perturbations, in which $v_t = [v_1 \;\; v_2]^T$ is the vector of perturbations in the rigid-body translations, $v_r = v_3$ is the perturbation in the rigid-body rotation, and $v_e = [v_4 \;\; v_5 \;\cdots\; v_{N+3}]^T$ is an N vector of pseudomodal coordinates corresponding to the elastic motions. Moreover, $T = [T_1 \;\; T_2 \;\; T_3 \;\; T_e^T]^T$ is the control vector, in which T_1 and T_2 are control forces acting at the mass center of the undeformed structure, where the mass center is assumed to lie on the rigid hub, T_3 is a torque acting on the rigid hub, and $T_e = [T_4 \;\; T_5 \;\cdots\; T_{p+3}]^T$ is a p vector of control torques acting on the flexible appendage. In addition, y_d and y_v represent the perturbed displacement and velocity measurement vectors, respectively, where the perturbed motion represents the difference between the sensor data and the corresponding commanded motion. We assume that there are two pairs of translational displacement and velocity sensors and one pair of angular displacement and angular velocity sensors collocated with the actuators T_1, T_2, and T_3. We assume further that there are $m/2$ pairs of angular displacement and velocity sensors located throughout the flexible appendage. Moreover, $\Omega_0(t)$ and $\dot{\Omega}_0(t)$ are the angular velocity and acceleration of the hub, $\Lambda = \mathrm{diag}(0 \;\; 0 \;\; 0 \;\; \omega_4^2 \;\; \omega_5^2 \;\cdots\; \omega_{N+3}^2)$ is the diagonal matrix of eigenvalues of the nonmaneuvering undamped structure, where ω_i are distinct natural frequencies of the structure and $\bar{D}_e = \mathrm{diag}(0 \;\; 0 \;\; 0 \;\; 2\zeta_4\omega_4 \;\; 2\zeta_5\omega_5 \;\cdots\; 2\zeta_{N+3}\omega_{N+3})$ is the damping matrix of the structure, in which ζ_i $(i = 4,5,\ldots,N+3)$ are small damping factors. Finally, the matrices $\bar{G}$, K, E, and C have the following form

$$\bar{G} = \left[\begin{array}{ccc|c} 0 & -1 & 0 & \\ 1 & 0 & 0 & 0_{3\times N} \\ 0 & 0 & 0 & \\ \hline & 0_{N\times 3} & & 0_{N\times N} \end{array}\right], \qquad \bar{D} = \left[\begin{array}{c|c} 0_{3\times 3} & 0_{3\times N} \\ \hline 0_{N\times 3} & \bar{D}_e \end{array}\right]$$

$$\bar{K} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0_{3\times N} \\ 0 & 0 & 0 & \\ \hline & 0_{N\times 3} & & \bar{K}_{22} \end{array}\right] \quad \text{(2a)}$$

$$\bar{E} = \left[\begin{array}{ccc|c} e_1 & 0 & 0 & 0_{2\times p} \\ 0 & e_1 & 0 & \\ 0 & 0 & e_2 & e_2\mathbf{1}_p^T \\ \hline \mathbf{0}_N & \boldsymbol{E}_2 & \boldsymbol{E}_3 & \bar{E}_e \end{array}\right]$$

$$\bar{C} = \left[\begin{array}{ccc|c} e_1 & 0 & 0 & \mathbf{0}_{1\times N} \\ 0 & e_1 & 0 & \boldsymbol{E}_2^T \\ 0 & 0 & e_2 & \boldsymbol{E}_3^T \\ \hline \mathbf{0}_{m/2} & \mathbf{0}_{m/2} & e_2\mathbf{1}_{m/2} & \bar{C}_e \end{array}\right] \quad \text{(2b)}$$

where e_1, e_2, $\bar{K}_{22}$, $\bar{E}_e$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$, $\bar{C}_e$ and Ψ are quantities depending on the system parameters.[5] The eigenvalues of the nonmaneuvering model are those of a beam with a mass at one end and free at the other end. As the eigenvalues increase, the effect of $\Omega_0^2(t)$ diminishes, so that the higher states behave as if the structure were not slewing. Furthermore, the inertia and moment of inertia of the hub are relatively large, so that the components of $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are relatively small. In fact, for the example considered in this paper, $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are two orders of magnitude smaller than the columns of $\bar{E}_e$ (see Ref. 14).

Equations (1) describe a linear time-varying system subjected to persistent disturbances. The equations are coupled by the control forces and torques and the measurements. The design of a reduced-order compensator, according to methods yet to be described (Sec. V), can be based on this set of equations. It turns out, however, that the translational equation precludes the achievement of quasiconstant gain matrices. Hence, to achieve suboptimal steady-state disturbance accommodation, it is necessary to control the translational motion independently. Moreover, to minimize the control gains so as to reduce spillover effects, it is desirable to control the rotational motion independently as well. To this end, if we let

$$T_3 = T_r - \sum_{j=4}^{p+3} T_j$$

and consider Eqs. (2), Eqs. (1) can be rewritten as

$$\ddot{v}_t + 2\Omega_0(t)P\dot{v}_t + [\dot{\Omega}_0(t)P - \Omega_0^2(t)I]v_t = e_1[T_1 \;\; T_2]^T \quad \text{(3a)}$$

$$\ddot{v}_r = e_2 T_r \quad \text{(3b)}$$

$$\ddot{v}_e + \bar{D}_e\dot{v}_e + [\Lambda_e - \Omega_0^2(t)\bar{K}_{22}]v_e = \bar{E}_e^* T_e + T_2\boldsymbol{E}_2 + T_r\boldsymbol{E}_3 - \dot{\Omega}_0(t)\Psi \quad \text{(3c)}$$

$$y_{1d} = e_1 v_1, \qquad y_{1v} = e_1\dot{v}_1 \quad \text{(3d)}$$

$$y_{2d} = e_1 v_2 + \boldsymbol{E}_2^T v_e, \qquad y_{2v} = e_1\dot{v}_2 + \boldsymbol{E}_2^T\dot{v}_e \quad \text{(3e)}$$

$$y_{3d} = e_2 v_3 + \boldsymbol{E}_3^T v_e, \qquad y_{3v} = e_2\dot{v}_3 + \boldsymbol{E}_3^T\dot{v}_e \quad \text{(3f)}$$

$$y_{ed} = \bar{C}_e^* v_e, \qquad y_{ev} = \bar{C}_e^*\dot{v}_e \quad \text{(3g)}$$

where P is a 2×2 skew symmetric unit matrix and

$$\bar{E}_e^* = \bar{E}_e - \boldsymbol{E}_e\mathbf{1}^T, \qquad \bar{C}_e^* = \bar{C}_e - \mathbf{1}\boldsymbol{E}_3^T \quad \text{(4)}$$

in which $\mathbf{1}^T = [1\ 1\ \cdots\ 1]$. Clearly, the rigid-body perturbations $\boldsymbol{v}_t$ and v_r are not subject to input disturbances caused by the maneuver, nor are they coupled through the coefficient matrix, and they can be controlled independently of the elastic motions. The control of the rigid-body perturbations is carried out according to a finite-time optimal control, such as one encountered in a linear quadratic regulator (LQR). In this design process, the rigid-body perturbations are extracted from the solution of the algebraic equations (1b), in which $\bar{C}$ is truncated by retaining the first $3+m/2$ columns only. Hence, the "measurement error" arising from the coupling terms

$$\sum_{i=1+m/2}^{N} E_{2i}v_{ei}, \qquad \sum_{i=1+m/2}^{N} E_{3i}v_{ei}$$

$$\sum_{i=1+m/2}^{N} \bar{E}_{2i}\dot{v}_{ei}, \qquad \sum_{i=1+m/2}^{N} \bar{E}_{3i}\dot{v}_{ei}$$

is due to relatively high elastic states, which are more difficult to excite. Moreover, the coupling terms $\bar{\boldsymbol{E}}_2$ and $\bar{\boldsymbol{E}}_3$ are usually very small,[14] so that the terms contaminating the output of the rigid-body sensors are relatively small and can be ignored in the control design process. We design the control gains so as to produce fast decay in $\boldsymbol{v}_t$ and v_r, which implies that the controls T_1, T_2, and T_r are of relatively short duration. Hence, because the terms $T_2\bar{\boldsymbol{E}}_2$ and $T_r\bar{\boldsymbol{E}}_3$ represent small transient disturbances, design of controls for the rigid-body perturbations and the elastic motions can be carried out independently. In view of this, we focus our attention on the control of the elastic motions under the persistent disturbances caused by the maneuver. Of course, the small coupling terms $T_2\bar{\boldsymbol{E}}_2$ and $T_r\bar{\boldsymbol{E}}_3$ are included in computer simulations, although their effect is imperceptible.

III. State Equations for the Elastic Motion

Introducing the state vector $\boldsymbol{z}_e(t) = [\boldsymbol{v}_e^T(t) \,|\, \dot{\boldsymbol{v}}_e^T(t)]^T$, as well as the coefficient matrices

$$A(t) = \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda_e + \Omega_0^2(t)\bar{K}_{22} & -\bar{D}_e \end{array}\right] \tag{5a}$$

$$B = \left[\begin{array}{c} 0 \\ \hline \bar{E}_e^* \end{array}\right] \tag{5b}$$

and the vectors

$$\boldsymbol{R} = [\boldsymbol{0}^T \,|\, \boldsymbol{\Psi}^T]^T \tag{6a}$$

$$\boldsymbol{d}(t) = [\boldsymbol{0}^T \,|\, (T_2\bar{E}_2 + T_r E_3)^T]^T \tag{6b}$$

we can rewrite Eq. (3c) in the state form

$$\dot{\boldsymbol{z}}_e = A(t)\boldsymbol{z}_e + B\boldsymbol{T}_e - \dot{\Omega}_0\boldsymbol{R} + \boldsymbol{d}(t) \tag{7}$$

Moreover, we write the output equation, Eq. (3g), as

$$\boldsymbol{y}_m = C\boldsymbol{z}_e \tag{8}$$

where

$$C = \left[\begin{array}{c|c} \bar{C}_e^* & 0 \\ \hline 0 & \bar{C}_e^* \end{array}\right] \tag{9}$$

It should be noted that $\Omega_0(t)$ and $\dot{\Omega}_0(t)$ are commanded angular velocity and acceleration, respectively, so that they are known a priori. For a minimum-time maneuver, the angular acceleration is according to a bang-bang control law. Hence, assuming ideal actuators, we have

$$\dot{\Omega}_0 = \begin{cases} c \text{ for } t_0 \le t < t_1 \\ -c \text{ for } t_1 \le t \le t_f \end{cases} \tag{10a}$$

$$\Omega_0(t) = \begin{cases} c(t-t_0) \text{ for } t_0 \le t < t_1 \\ -c(t-t_f) \text{ for } t_1 \le t \le t_f \end{cases} \tag{10b}$$

where c is a constant, t_0 and t_f are the initial and final time of the maneuver, respectively, and $t_1 = (t_f - t_0)/2$ is the switching time. On the other hand, we still regard $\boldsymbol{\Psi}$ as insufficiently accurate for direct compensation. Hence, we shall treat the persistent disturbance $-\dot{\Omega}_0(t)\boldsymbol{\Psi}$ as unknown, except that it is piecewise constant. Moreover, we assume that $A(t)$, as given by Eq. (5), is a quasicontractively stable matrix (see Definition 2 in the Appendix).

IV. Reduced-Order Model

The actual elastic displacement, which implies an infinite number of degrees of freedom, is approximated by a finite-dimensional vector. It is typical of structures that computed higher modes are inaccurate.[2] Moreover, inherent damping improves the robustness of higher modes and the components of the disturbance vector $-\dot{\Omega}_0(t)\boldsymbol{\Psi}$ corresponding to the higher states tend to have lower values. Hence, it is reasonable to retain the first N components of $\boldsymbol{v}_e$ in the perturbed elastic model. The state representation of this model is defined by Eq. (7). We denote the order of the elastic model by n_M, where $n_M = 2N$, and refer to it as the "modeled system." It is assumed that n_M is sufficiently high that it is unlikely that the ignored degrees of freedom will destabilize the actual closed-loop system and will participate significantly in the motion.

Control implementation considerations dictate further truncation. Moreover, the spillover effect can be alleviated by shifting the observation spillover to more robust parts of the model.[19] Hence, we retain the first $n_C/2$ components of $\boldsymbol{v}_e$ for control and estimate $n_O/2$ additional states without controlling them. Hence, $\boldsymbol{z}_e = [\boldsymbol{z}_C^T \,|\, \boldsymbol{z}_O^T \,|\, \boldsymbol{z}_R^T]^T$ and $n_C + n_O + n_R = n_M$. Considering these definitions and dividing the control $\boldsymbol{T}_e$ into the feedback control $\boldsymbol{u}_C$ and the disturbance accommodation control $\boldsymbol{u}_D$, Eqs. (7) and (8) can be partitioned as

$$\begin{bmatrix} \dot{\boldsymbol{z}}_C \\ \dot{\boldsymbol{z}}_O \\ \dot{\boldsymbol{z}}_R \end{bmatrix} = \left[\begin{array}{c|c|c} A_{CC} & A_{CO} & A_{CR} \\ \hline A_{OC} & A_{OO} & A_{OR} \\ \hline A_{RC} & A_{RO} & A_{RR} \end{array}\right] \begin{bmatrix} \boldsymbol{z}_C \\ \boldsymbol{z}_O \\ \boldsymbol{z}_R \end{bmatrix} + \begin{bmatrix} B_C \\ B_O \\ B_R \end{bmatrix} (\boldsymbol{u}_C + \boldsymbol{u}_D) - \dot{\Omega}_0 \begin{bmatrix} \boldsymbol{R}_C \\ \boldsymbol{R}_O \\ \boldsymbol{R}_R \end{bmatrix} + \begin{bmatrix} \boldsymbol{d}_C(t) \\ \boldsymbol{d}_O(t) \\ \boldsymbol{d}_R(t) \end{bmatrix} \tag{11a}$$

$$\boldsymbol{y}_m = [C_C \,|\, C_O \,|\, C_R] \begin{bmatrix} \boldsymbol{z}_C \\ \boldsymbol{z}_O \\ \boldsymbol{z}_R \end{bmatrix} \tag{11b}$$

where

$$A_{ii} = \left[\begin{array}{c|c} 0_{i\times i} & I_{i\times i} \\ \hline -\Lambda_{ii} + \Omega_0^2(t)\bar{K}_{ii} & -\bar{D}_{ii} \end{array}\right] \tag{12a}$$

$$A_{ij} = \Omega_0^2(t)\left[\begin{array}{c|c} 0_{i\times j} & 0_{i\times j} \\ \hline \bar{K}_{ij} & 0_{i\times j} \end{array}\right], \qquad i,j = C,O,R \tag{12b}$$

V. Design of a Reduced-Order Compensator

Our objective is to design a control law for the controlled part of the model according to a quadratic performance measure such that the response to any initial condition will decay below an arbitrary small number within the finite-time interval. We refer to a system that can be stabilized according to this definition as strictly finite-time stabilizable (see Definition 3 in the Appendix).

Due to the nature of the persistent disturbances, as inferred from Eqs. (10), we propose to carry out the control and estimation over one-half of the maneuver period at a time. Hence, we denote the control and estimation interval by

$\tau = [t_i, t_h)$, where $t_i = t_0$ and $t_h = t_1$ for the first half of the maneuver period and $t_i = t_1$ and $t_h = t_f$ for the second half.

A. Controller Design

To design the feedback control for the stabilization of the reduced-order model, we consider the controlled model in the absence of disturbances. From Eqs. (11), we can write

$$\dot{z}_C(t) = A_{CC}(t) z_C(t) + B_C u_C(t), \qquad z_C(t_i) = z_{Ci}, \qquad t \in \tau \tag{13}$$

In view of Eq. (10b), we conclude that $A_{CC}(t)$ and B_C are analytic matrices on τ and are bounded for all $t \in \tau$. The dimension of z_C is n_C and that of u_C is p, where $p \le n_C/2$. It is shown in Ref. 17 that the pair $[A_{CC}(t), B_C]$ is completely controllable in every arbitrary subinterval of τ (i.e., totally controllable[20] for every $t \in \tau$). To achieve the control objective described here, we consider the performance measure

$$J = \int_{t_i}^{t_h} e^{2\alpha t}(z_C^T Q z_C + u_C^T R u_C)\, dt + e^{2\alpha t_h} z_C^T(t_h) S_1 z_C(t_h) \tag{14}$$

where $\alpha \ge 0$ and $Q = Q^T > 0$, $R = R^T > 0$, $S_1 = S_1^T \ge 0$. The problem defined by Eqs. (13) and (14) can be reduced to a standard LQR problem by introducing the transformations[21]

$$z_C^*(t) = e^{\alpha t} z_C(t) \tag{15a}$$

$$u_C^*(t) = e^{\alpha t} u_C(t) \qquad \alpha \ge 0 \tag{15b}$$

Inserting Eqs. (15) into Eqs. (13) and rearranging, we obtain

$$\dot{z}_C^*(t) = A_{CC}^*(t) z_C^*(t) + B_C u_C^*(t) \tag{16}$$

where

$$A_{CC}^*(t) = A_{CC}(t) + \alpha I \tag{17}$$

It is shown in Ref. 17 that the total controllability of the original system, Eqs. (13), also implies the total controllability of the system defined by Eq. (16). The control law minimizing J is

$$u_C(t) = K_C(t) z_C(t) \tag{18a}$$

where

$$K_C = -R^{-1} B_C^T S(t) \tag{18b}$$

in which $S(t)$ is the solution of the matrix Riccati equation

$$\dot{S}(t) = -Q - S(t) A_{CC}^*(t) - A_{CC}^{*T}(t) S(t) + S(t) B_C R^{-1} B_C^T S(t), \qquad S(t_h) = S_1 \tag{19}$$

Next, we wish to develop an explicit sufficiency condition guaranteeing the strictly finite-time stabilizability of the closed-loop system. To this end, we consider the sufficiency condition for strictly finite-time stability (see the Appendix) and define the Lyapunov function

$$V(z_C^*, t) = z_C^{*T} S(t) z_C^* \tag{20}$$

Differentiating Eq. (20) with respect to time, evaluating the result along solutions of Eq. (16), and considering Eq. (19), we obtain

$$\dot{V}(z_C^*, t) = -z_C^{*T} C(t) z_C^* \tag{21}$$

where

$$C(t) = Q + S(t) B_C^T R^{-1} B_C S(t) \tag{22}$$

is a positive definite matrix for all $t \in \tau$, because $Q > 0$ and $S(t) B_C^T R^{-1} B_C S(t) \ge 0$ for all $t \in \tau$. This proves the first part of the sufficiency condition. The second part of the sufficiency condition, Eq. (A12), is satisfied if[17]

$$\frac{\lambda_M[S(t_i, \alpha + \Delta\alpha)]}{\lambda_M[S(t_i, \alpha)]} < \exp[2\Delta\alpha(t - t_i)] \text{ for all } t \in (T_1, t_h)$$

$$t_i < t_1 < t_h, \qquad \alpha \ge \bar{\alpha}(\varepsilon) \ge 0, \qquad \Delta\alpha > 0 \tag{23}$$

and we recognize that this condition is expressed in terms of the Riccati solution $S(t)$ evaluated at the initial time t_i, where the solution is likely to reach its steady-state value if α is sufficiently large. Hence, we can finally state that, if inequality (23) is satisfied, then the system represented by Eqs. (13), (18), and (19) is strictly finite-time stabilizable. Hence, to every ε and Z_{Ci} there corresponds an $\bar{\alpha}(\varepsilon, Z_{Ci})$ such that for every $\alpha > \bar{\alpha}(\varepsilon, Z_{Ci})$ the response to Z_{Ci} decays to a value below ε within the finite-time interval τ.

B. Observer Design

The observer is expected to estimate not only part of the state vector but also part of the persistent disturbance vector. Hence, let us introduce the notation

$$-\dot{\Omega}_0 R = W = [W_C^T \,|\, W_O^T \,|\, W_R^T]^T \tag{24}$$

where W_C, W_O, and W_R are the parts of $-\dot{\Omega}_0 R$ corresponding to z_C, z_O, and z_R, respectively. Considering the disturbance dynamics, Eq. (10a) and the structure of R Eq. (6a), we denote the part of $-\dot{\Omega}_0 \Psi$ to be estimated by f_E and introduce the relations

$$\dot{f}_E = \delta_E(t) \tag{25a}$$

$$W_E = H_E f_E \tag{25b}$$

where f_E is an $n_W/2$ vector, $n_W \le n_C + n_O$, $\delta_E(t)$ is a vector of Dirac delta functions of the same dimension, and H_E is an $(n_C + n_O) \times n_W/2$ matrix of zeros and ones. For convenience, we rewrite Eq. (24) as

$$-\dot{\Omega}_0 R = [(W_E^{**})^T \,|\, W_R^T]^T + \begin{bmatrix} H_E \\ 0 \end{bmatrix} f_E \tag{26}$$

where 0 denotes a null matrix. Defining the vector to be estimated as $\eta_E = [z_C^T \,|\, z_O^T \,|\, f_E^T]^T$, Eqs. (11), (25), and (26) can be combined into

$$\begin{bmatrix} \dot{\eta}_E \\ \dot{z}_R \end{bmatrix} = \begin{bmatrix} A_{EE} & A_{ER} \\ A_{RE} & A_{RR} \end{bmatrix} \begin{bmatrix} \eta_E \\ z_R \end{bmatrix} + \begin{bmatrix} B_E \\ B_R \end{bmatrix} (u_C + u_D) + \begin{bmatrix} I_E^* \\ 0 \end{bmatrix} \delta_E(t) + \begin{bmatrix} W_E^* \\ W_R \end{bmatrix} + \begin{bmatrix} d_E^*(t) \\ d_R(t) \end{bmatrix} \tag{27}$$

where

$$A_{EE} = \begin{bmatrix} A_{CC} & A_{CO} & \\ A_{OC} & A_{OO} & H_E \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{ER} = \begin{bmatrix} A_{CR} \\ A_{OR} \\ 0 \end{bmatrix}, \quad A_{RE} = [A_{RC} \,|\, A_{RO} \,|\, 0]$$

$$B_E = \begin{bmatrix} B_C \\ B_O \\ 0 \end{bmatrix}, \quad I_E^* = \begin{bmatrix} 0 \\ 0 \\ I_E \end{bmatrix}, \quad W_E^* = \begin{bmatrix} W_E^{**} \\ 0 \end{bmatrix}, \quad d_E^* = \begin{bmatrix} d_C \\ d_O \\ 0 \end{bmatrix} \tag{28}$$

in which I_E is an $n_W/2 \times n_W/2$ identity matrix; the various submatrices in Eqs. (28) can be deduced from Eqs. (5). Consistent with the preceding, the output vector can be written in the form

$$y_m = C_E \eta_E + C_R z_R \tag{29}$$

where, in accordance with Eq. (9),

$$C_E = \begin{bmatrix} \bar{C}_C^* & 0 & \bar{C}_O^* & 0 & 0 \\ 0 & \bar{C}_C^* & 0 & \bar{C}_O^* & 0 \end{bmatrix}, \quad C_R = \begin{bmatrix} \bar{C}_R^* & 0 \\ 0 & \bar{C}_R^* \end{bmatrix} \tag{30}$$

Introducing the estimated vector $\hat{\eta}_E = [\hat{z}_C^T \,|\, \hat{z}_O^T \,|\, \hat{f}_E^T]^T$, we consider an observer described by

$$\dot{\hat{\eta}}_E = A_{EE}\hat{\eta}_E + B_E(u_C + u_D) + K_E(\hat{y}_m - y_m) \tag{31}$$

where $K_E = K_E(t)$ is the observer gain matrix and $\hat{y}_m = C_E\hat{\eta}_E$. Defining the observer error vector as

$$e = \eta_E - \hat{\eta}_E \tag{32}$$

we obtain the observer error equation

$$\dot{e}(t) = [A_{EE}(t) + K_E(t)C_E]e(t) + [A_{ER}(t) + K_E(t)C_R]z_R(t)$$
$$+ I_E^*\delta_E(t) + W_E^* + d_E^*(t) \tag{33}$$

It is shown in Ref. 17 that the pair $[A_{EE}(t), C_E]$ is completely observable in every subinterval of τ (i.e., totally observable[20] for every $t \in \tau$). Introducing the transformation

$$e^*(t) = e^{\beta t}e(t), \qquad \beta \geq 0 \tag{34}$$

and inserting Eq. (34) into Eq. (33) and retaining the homogeneous part, we obtain

$$\dot{e}^*(t) = [A_{EE}(t) + \beta I + K_E(t)C_E]e^*(t) \tag{35}$$

Following Ref. 22, we propose the following gain matrix

$$K_E(t) = -\tfrac{1}{2}P(t)C_E^T \tag{36}$$

where $P(t)$ is the solution of the Riccati equation

$$\dot{P} = (A_{EE} + \beta I)P + P(A_{EE}^T + \beta I) - PC_E^TC_EP + Q_E$$
$$P(t_i) = P_0 \tag{37a}$$

where

$$Q_E = Q_E^T > 0, \qquad P_0 = P_0^T > 0 \tag{37b}$$

Next, we wish to develop a sufficiency condition guaranteeing the strictly finite-time stabilizability of the system defined by Eqs. (35–37). To this end, we define the Lyapunov function

$$V(e^*, t) = e^{*T}P^{-1}(t)e^* \tag{38}$$

Then, by a completely dual process to the one described in Sec. V.A, it is shown in Ref. 17 that the positive definiteness of Q_E and the condition

$$\frac{\lambda_M[P(t, \beta + \Delta\beta)]}{\lambda_M[P(t,\beta)]} < \exp[2\Delta\beta(t - t_1)]$$

$$\text{for all } t \in (T_1, t_h), \quad t_i < T_1 < t_h, \quad \beta \geq \bar{\beta}(\varepsilon) \geq 0, \quad \Delta\beta > 0 \tag{39}$$

are sufficient to guarantee a strictly finite-time stabilizable system. Hence, to every ε and $e(t_i)$ corresponds a $\bar{\beta}[\varepsilon, e(t_i)]$ such that for every $\beta > \bar{\beta}[\varepsilon, e(t_i)]$ the response of the system to $e(t_i)$ decays to a value below ε within the finite-time interval τ.

Next, we wish to derive the closed-loop equations for the modeled system incorporating the observer just described. To this end, we write the dynamic control and the persistent disturbance control in the form

$$u_C(t) = K_C(t)\hat{z}_C(t), \qquad u_D(t) = K_D\hat{f}_E(t) \tag{40}$$

Then, denoting the uncontrolled state vector by $z_U = [z_O^T \,|\, z_R^T]^T$ and considering Eqs. (27) and (31), we obtain

$$\begin{bmatrix} \dot{z}_C \\ \dot{e} \\ \dot{z}_U \end{bmatrix} = \begin{bmatrix} \tilde{A}_{CC} & \tilde{A}_{CE} & \tilde{A}_{CU} \\ 0 & \tilde{A}_{EE} & \tilde{A}_{EU} \\ \tilde{A}_{UC} & \tilde{A}_{UE} & \tilde{A}_{UU} \end{bmatrix} \begin{bmatrix} z_C \\ e \\ z_U \end{bmatrix} + \begin{bmatrix} W_C + B_CK_Df_E \\ W_E^* \\ W_U + B_UK_Df_E \end{bmatrix} + \begin{bmatrix} 0 \\ I_E \\ 0 \end{bmatrix}\delta_E(t) + \begin{bmatrix} d_C(t) \\ d_E^*(t) \\ d_U(t) \end{bmatrix} \tag{41}$$

where

$$\tilde{A}_{CC} = A_{CC} + B_CK_C, \quad \tilde{A}_{CE} = -[B_CK_C \,|\, 0 \,|\, B_CK_D]$$

$$\tilde{A}_{CU} = [A_{CO} \,|\, A_{CR}], \quad \tilde{A}_{EE} = A_{EE} + K_EC_E,$$

$$\tilde{A}_{EU} = \begin{bmatrix} 0 & A_{CR} + K_{EC}C_R \\ 0 & A_{OR} + K_{EO}C_R \\ 0 & K_{Ef}C_R \end{bmatrix}, \quad \tilde{A}_{UC} = \begin{bmatrix} A_{OC} + B_OK_C \\ A_{RC} + B_RK_C \end{bmatrix}$$

$$\tilde{A}_{UE} = \begin{bmatrix} B_OK_C & 0 & B_OK_D \\ B_RK_C & 0 & B_RK_D \end{bmatrix}, \quad \tilde{A}_{UU} = \begin{bmatrix} A_{OO} & A_{OR} \\ A_{RO} & A_{RR} \end{bmatrix} \tag{42}$$

in which we introduced the notation

$$K_E = [K_{EC}^T \,|\, K_{EO}^T \,|\, K_{Ef}^T]^T, \quad B_U = \begin{bmatrix} B_O \\ B_R \end{bmatrix}$$

$$W_U = \begin{bmatrix} W_O \\ W_R \end{bmatrix}, \quad d_U = \begin{bmatrix} d_O \\ d_R \end{bmatrix} \tag{43}$$

In future discussions, we will refer to the coefficient matrix in Eq. (41) as $\tilde{A}(t)$.

VI. Disturbance Accommodation

The goal of disturbance accommodation is to minimize the response of an output vector associated with the disturbance. To this end, we define an n_M-dimensional output vector

$$y_l(t) = L_yz_e(t) \tag{44}$$

For the minimization process contemplated, we rearrange the state vector in Eq. (41) and retain only the terms related to the observed disturbance vector, $\hat{f}_E(t)$. Hence, we rewrite Eq. (41) in the form

$$\begin{bmatrix} \dot{z}_e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z_e \\ e \end{bmatrix} + \begin{bmatrix} H_F + BK_D \\ 0 \end{bmatrix} \hat{f}_E \tag{45}$$

where

$$F_{11} = \begin{bmatrix} \tilde{A}_{CC} & \tilde{A}_{CU} \\ \tilde{A}_{UC} & \tilde{A}_{UU} \end{bmatrix}, \quad F_{12} = [-BK_C \,|\, 0 \,|\, H_F],$$

$$F_{21} = [0, \,|\, \tilde{A}_{EU}], \quad F_{22} = \tilde{A}_{EE}, \quad H_F = \begin{bmatrix} H_E \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_C \\ B_U \end{bmatrix} \tag{46}$$

Then, denoting by $y_f(t)$ the response to $\hat{f}_E(t)$ we obtain

$$y_f(t) = \int_{t_i}^{t} [L_y \,|\, 0]\Phi_F(t,\sigma)\begin{bmatrix} H_F + BK_D \\ 0 \end{bmatrix} \hat{f}_E(\sigma)\, d\sigma, \quad t \in \tau \tag{47}$$

where $\Phi_F(t,\sigma)$ is the transition matrix corresponding to the coefficient matrix $F(t)$ in Eq. (45). To comply with on-line computer limitations, we seek a constant gain matrix K_D minimizing the performance measure

$$J_D = \lim_{t \to t_h} \|y_f(t)\|^2 \tag{48}$$

The solution of this problem has meaning only if quasi-time-invariant conditions exist for $t \to t_h$. Hence, we assume now, and justify later, that the coefficient matrix is such that

$$F(t) = F_0 + \varepsilon F_1(t), \qquad \varepsilon \ll 1, \qquad t \in \tau \tag{49}$$

where F_0 is a constant matrix. Then, the transition matrix can be shown to have the form[23]

$$\Phi_F(t,t_i) = e^{F_0(t-t_i)} + \varepsilon e^{F_0 t} \int_{t_i}^{t} e^{-F_0\tau} F_1(\tau) e^{F_0\tau}\, d\tau + O(\varepsilon^2) \tag{50}$$

Using Eqs. (47) and (50), retaining the zero-order term only and denoting the constant part of $\hat{f}_E(t)$ by $\hat{f}_e$, we obtain

$$y_{f0}(t) = -[L_y \mid 0] F_0^{-1} [I - e^{F_0(t-t_i)}] \left|\begin{array}{c} H_F + BK_D \\ \hline 0 \end{array}\right| \hat{f}_E, \qquad t \in \tau \tag{51}$$

At this point, we assume that F_0 is a stable matrix, i.e.,

$$Re\, \lambda_i(F_0) < 0, \quad i = 1,2,\ldots, n_A; \quad n_A = n_M + n_C + n_O + N_W/2 \tag{52}$$

which guarantees the existence of F_0^{-1}. Next, let us denote by $\bar{y}_{f0}$ the constant part of $y_{f0}(t)$ and determine the constant gain matrix K_D, that minimizes the performance measure

$$\bar{J}_D = \|\bar{y}_{f0}\|^2 \tag{53}$$

The transient part of $y_{f0}(t)$ and of $\hat{f}_E(t)$, which are not included in the minimization process, decay below small threshold values as $t \to t_h$ provided $\tilde{A}(t)$ is quasicontractively stable, such that $\rho \ll 1$ (see Definition 2 in the Appendix), and

$$\lim_{t \to t_h} \|e^{F_0 t}\| \ll 1 \tag{54}$$

To produce the gain matrix K_D, we partition F_0 in the same way as in Eq. (45) and introduce the notation

$$F_0 = \left[\begin{array}{c|c} \bar{F}_C & \bar{F}_{12} \\ \hline \bar{F}_{21} & \bar{F}_E \end{array}\right] \tag{55}$$

Inserting Eq. (55) into Eq. (51), we obtain

$$\bar{y}_{f0} = -L_y \Gamma^{-1}(H_F + BK_D)\bar{f}_E = -L_y\Gamma^{-1}(H_F\bar{f}_E + Bu_D) \tag{56}$$

where

$$\Gamma = \bar{F}_C = -\bar{F}_{12}\bar{F}_E^{-1}\bar{F}_{21} \tag{57}$$

This requires that $\bar{F}_E$ be nonsingular, which implies the assumption

$$Re\, \lambda_i(\bar{F}_E) < 0, \qquad i = 1,2,\ldots, n_C + n_O + n_{W/2} \tag{58}$$

In addition, Γ itself must be nonsingular. Using Eqs. (55), (57), and (58), it can be shown easily that $\Gamma^{-1} = (F_0^{-1})_{11}$, where F_0^{-1} is partitioned in the same way as F_0 in Eq. (55). At this point, we seek the value of u_D that minimizes $\bar{J}_D$. This value must satisfy

$$\frac{\partial \bar{J}_D}{\partial u_D} = B^T\Gamma^{-T}W\Gamma^{-1}(H_F\bar{\bar{f}}_E + Bu_D) = 0 \tag{59}$$

where $\Gamma^{-T} = (\Gamma^{-1})^T$ and $W = L_y^T L_y$, so that, recalling Eq. (40), the minimizing gain matrix has the expression

$$K_D = -(B^T\Gamma^{-T}W\Gamma^{-1}B)^{-1}B^T\Gamma^{-T}W\Gamma^{-1}H_F \tag{60}$$

A sufficient condition for the existence of this minimum is

$$B^T\Gamma^{-T}W\Gamma^{-1}B > 0 \tag{61}$$

Considering inequalities (52), this condition is guaranteed by choosing $W > 0$ and by placing the actuators so as to ensure a full-rank matrix B.

In the following, we propose to validate two of the necessary conditions implied by Eq. (60), namely assumptions (49) and (58). The other necessary condition is assumption (52), which represents a design requirement and is analyzed in the next section.

Let us consider the system described by Eqs. (13), (18), and (19). The discussion can be generalized to some extent by replacing the matrix $A_{CC}^*(t)$, Eq. (17), by

$$A^*(t) = \left[\begin{array}{c|c} \alpha I & I \\ \hline \varphi(t)\bar{K}_{22} - \Lambda_e & \alpha I - \bar{D}_e \end{array}\right] \tag{62}$$

where $0 \le \alpha \le \alpha_M$ and $\varphi(t) = \varphi_0 t^k$, $k \ge 2$. We assume that φ_0 is such that $\varphi(t)\|\bar{K}_{22}\| \gg \|\Lambda_e\|$ for $t \in \tau$, where t is sufficiently large. As in the case of the original matrix $A_{CC}^*(t)$, the pair $[A^*(t), B_C]$ is totally controllable for every $t \in \tau$. Then, it is verified numerically that the Riccati matrix $S(t)$ possesses the following characteristics:

1) By increasing the parameter α, the steady-state solution of the Riccati equation, Eq. (19), can be reached within τ.

2) The previous steady-state solution can be expressed as

$$\bar{S}(t) = \bar{S}_0 + \varepsilon_1(\alpha)S_1(t) \tag{63}$$

where $\bar{S}_0$ is the solution of the algebraic Riccati equation using the matrix A_0^*, obtained from Eq. (62) by setting $\varphi(t) \equiv 0$, and $\varepsilon_1(\alpha)$ is a positive scalar decreasing monotonically with increasing α.

3) Choosing the end condition $S_1 = \bar{S}_0$ for Eq. (19), we obtain

$$S(t) = \bar{S}(t) + \varepsilon_2(\alpha)S_2(t), \qquad \text{for all } t \in \tau \tag{64}$$

where $\varepsilon_2(\alpha)$ is a positive scalar. Then, in view of Eq. (63),

$$S(t) = \bar{S}_0 + \varepsilon_3(\alpha)S_3(t), \qquad \text{for all } t \in \tau \tag{65}$$

where $\varepsilon_3(\alpha)$ decreases monotonically with increasing α.

These characteristics hold for the solution of the observer Riccati equation, Eq. (37a). Indeed, choosing the initial condition $P_i = \bar{P}_0$, we obtain

$$P(t) = \bar{P}_0 + \varepsilon_4(\beta)P_1(t), \qquad \text{for all } t \in \tau \tag{66}$$

where $\varepsilon_4(\beta)$ is a positive scalar decreasing monotonically with increasing β. We refer to solutions (65) and (66) as quasiconstant.

Next, let us introduce Eq. (65) into Eq. (18b) and obtain the quasiconstant optimal control gain matrix

$$K_C(t) = \bar{K}_C + \varepsilon K_{C1}(t), \qquad \varepsilon \ll 1 \tag{67}$$

where

$$\bar{K}_C = -R^{-1}B_C^T\bar{S}_0 \tag{68a}$$

$$K_{C1}(t) = -R^{-1}B_C^T S_1(t) \tag{68b}$$

Similarly, inserting Eq. (66) into Eq. (36), we obtain the quasiconstant observer gain matrix

$$K_E(t) = \bar{K}_E + \varepsilon K_{E1}(t), \qquad \varepsilon \ll 1 \tag{69}$$

where

$$K_E = -\tfrac{1}{2}\bar{P}_0 C_E^T \quad (70a)$$

$$K_{E1}(t) = -\tfrac{1}{2}P_1(t)C_E^T \quad (70b)$$

Introducing Eqs. (67) and (69) into Eqs. (46), recalling Eqs. (42) and (12b), and assuming that $\|A_{ij}(t)\|$ is small compared to $\|\bar{K}_C\|$, $\|\bar{K}_E\|$, and $\|\Lambda_{ei}\|$ $(i = 1 + n_C/2,\ 2 + n_C/2, \ldots, n_M/2)$, the factor ε in Eq. (49) must be regarded as small.

Next, we verify the assumption implied by inequalities (58). Introducing Eqs. (70) into the fourth of Eqs. (46) and recalling the fourth of Eqs. (42) we obtain

$$\bar{F}_E = \bar{A}_{EE} - \tfrac{1}{2}\bar{P}_0 C_E^T C_E \quad (71)$$

where $\bar{A}_{EE} = A_{EE}$ $(\Omega_0 \equiv 0)$ and $\bar{P}_0$ is the solution of the algebraic Riccati equation

$$(\bar{A}_{EE} + \beta I)\bar{P}_0 + \bar{P}_0(\bar{A}_{EE}^T + \beta I) - \bar{P}_0 C_E^T C_E \bar{P}_0 + Q_E = 0 \quad (72)$$

Inserting $\bar{A}_{EE}$ from Eq. (71) into Eq. (72), we obtain the Lyapunov equation

$$(\bar{F}_E + \beta I)\bar{P}_0 + \bar{P}_0(\bar{F}_E^T + \beta I) = -Q_E \quad (73)$$

In view of the fact that $\bar{P}_0$ and Q_E are symmetric and positive definite, we have[24]

$$Re\,\lambda_i(\bar{F}_E + \beta I) < 0, \qquad i = 1,2,\ldots, n_C + n_O + n_{w/2} \quad (74)$$

VII. Finite-Time Stability of the Modeled System

The third stage of the design involves the determination of the optimal values for the convergence factors α and β so as to minimize the supremum time constant of the modeled system and at the same time to preserve the exponential stability of the matrix F_0 as to agree with assumption (52). The closed-loop modeled system is defined by Eq. (41). Introducing the η_A-dimensional modeled state vector $\eta_A = [z_C^T \,|\, e^T \,|\, z_U^T]^T$, we can rewrite the homogeneous part of Eq. (41) as

$$\dot{\eta}_A(t) = A(t)\eta_A(t), \qquad \eta_A(t_i) = \eta_{Ai} \text{ for all } t \in \tau \quad (75)$$

The design process is based on the fact that the system defined by Eq. (75), in which the values of α and β are the minimal values guaranteeing quasi-time-invariant optimal gains and proper dynamics for the closed-loop controlled model, is exponentially contractively stable (see Definition 1 in the Appendix). Indeed, the convergence characteristics of system (75), as well as those of F_0, are dominated by the main diagonal submatrices, where the dominance of the main diagonal submatrices can be enhanced by adding critical uncontrolled states to the observed model. The diagonal submatrices $\tilde{A}_{CC}(t)$ and $\tilde{A}_{EE}(t)$ of $\tilde{A}(t)$ are finite-time stable by design and the open-loop submatrix $\tilde{A}_{UU}(t)$ is finite-time stable by assumption (Sec. III). Similarly, the diagonal submatrices $\bar{F}_C$ and $\bar{F}_E$ of F_0 are exponentially stable.

Comparing the matrix F_0, Eq. (55), to the constant part of $\tilde{A}(t)$, denoted by $\tilde{A}_0$, it can be shown that the eigenvalues of these two matrices differ only slightly and that the difference can be reduced by increasing the order of the observed model.[17] Hence, the procedure minimizing the supremum time constant should also guarantee that

$$Re\,\lambda_i(\tilde{A}_0) < 0, \qquad i = 1,2,\ldots, n_A \quad (76)$$

To this end, we introduce the following Lyapunov function[25]

$$V(\eta_A, t) = \eta_A^T(t) B_a(t) \eta_A(t) \quad (77)$$

where $B_a(t)$ is a Hermitian matrix given by

$$B_a(t) = [U_a(t)U_a^H(t)]^{-1} \quad (78)$$

in which the superscript H denotes the complex conjugate transpose of a matrix and $U_a(t)$ is the matrix of instantaneous eigenvectors of $\tilde{A}(t)$. It is assumed that the eigenvalues of $\tilde{A}(t)$ are distinct for all $t \in \tau$, so that the matrix $U_a(t)$ is always nonsingular. It follows that

$$\dot{V}(\eta_A, t) = \eta_A^T(t) U_a^{-H}(t) C_a(t) U_a^{-1}(t) \eta_A(t) \quad (79)$$

where $[\ ]^{-H}$ is the inverse of $[\ ]^H$, and

$$C_a(t) = 2\,Re[\Lambda_a(t)] + U_a^H(t)\dot{B}_a(t)U_a(t) \quad (80)$$

in which $\Lambda_a(t)$ is the diagonal matrix of the eigenvalues of $\tilde{A}(t)$. Then, following Eqs. (A.14) and (A.15) in the Appendix, it can be shown[17] that if the conditions

$$Re\,\lambda_i[C_a(t)] < 0, \qquad i = 1,2,\ldots, n_A \text{ for all } t \in \tau \quad (81)$$

are preserved, then the supremum time constant of the modeled system is

$$\pi = 2\{\min_{t\in\tau} |\lambda_m[C_a(t)]|\}^{-1} \quad (82)$$

The minimum supremum time constant is given by

$$\pi_m = \min_{\alpha,\beta}[\pi(\alpha,\beta)] \quad (83)$$

We recognize that Eqs. (80) and (81) also imply condition (76).

VIII. Numerical Example

The above developments were applied to the flexible spacecraft shown in Fig. 1. The mathematical model consists of a rigid hub and a flexible appendage in the form of a uniform beam 24 ft long. The mass of the spacecraft is 34.08 lb s^2 ft^{-1} and the mass moment of inertia about the mass center is 264 lb s^2 ft. The perturbed model is assumed to possess two rigid-body translational, one rigid-body rotational, and seven elastic degrees of freedom. The first 10 natural frequencies of the structure are 0, 0, 0, 0.32, 1.8, 5.0, 9.8, 16.2, 24.2, and 33.8 Hz. All damping factors are assumed to have the same value $\zeta_i = 0.01$ $(i = 4,5,\ldots, 10)$.

The rigid-body slewing of the spacecraft is carried out according to a minimum-time control policy. The 180-deg maneuver is relatively fast, so that in the absence of vibration control the elastic deformations tend to be large, as shown in Fig. 2.

There are seven controlled degrees of freedom, three representing rigid-body motions and four representing elastic motions. As suggested in Sec. II, the rigid-body perturbations are

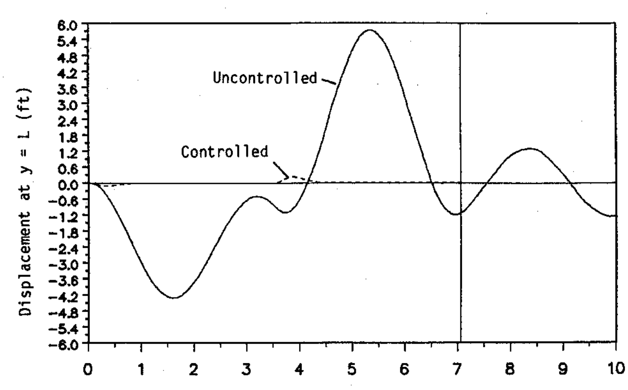

Fig. 2 Displacement at the tip; maneuver angle = 180 deg.

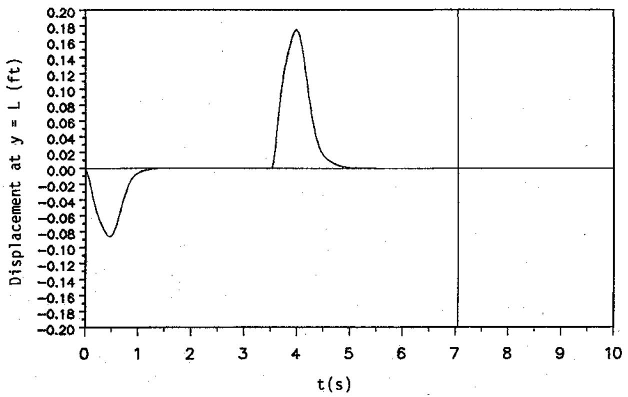

Fig. 3 **Displacement at the tip ($\alpha = 3$); maneuver angle = 180 deg, eighth-order model, eighth-order controller.**

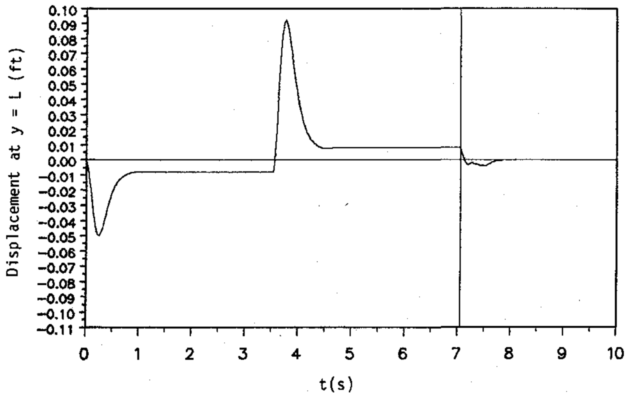

Fig. 6 Displacement at the tip ($p = 4$, $\alpha = 5$); maneuver angle = 180 deg, 14th-order model, eighth-order controlled model.

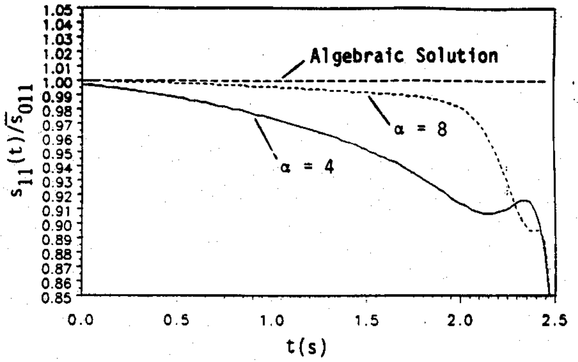

Fig. 4 Deviation of time-varying Riccati solution from algebraic solution for $S(t_f) = 0.85\bar{S}_0$.

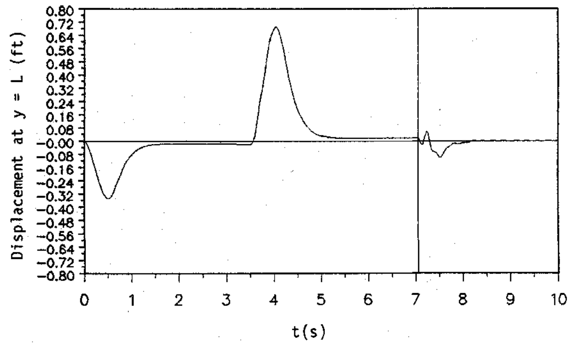

Fig. 7 Displacement at the tip ($p = 1$, $\alpha = 2$); maneuver angle = 180 deg, 14th-order model, eighth-order controlled model.

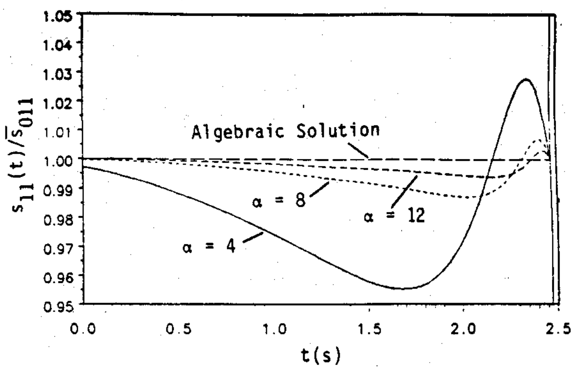

Fig. 5 Deviation of time-varying Riccati solution from algebraic solution for $S(t_f) = \bar{S}_0$.

controlled independently of the elastic motions by three sets of collocated actuators and sensors, one for each degree of freedom, located at the mass center of the spacecraft assumed to lie on the hub. The feedback control is applied during and after the maneuver, whereas the disturbance accommodation operates only during the maneuver. The observer is designed to estimate the controlled state and a two-dimensional disturbance vector. From now on, we focus our attention on the 14th-order modeled system, which consists of the elastic degrees of freedom, although the simulation includes all the degrees of freedom, i.e., it is based on the full set of Eqs. (3). In every case, the response represents the deflection at the tip of the beam, at which point the deflection tends to be the largest.

Figure 3 shows the response of an eight-order controlled model without the residual effect, where $m/2 = p = 4$. As $t \to t_h$, the displacement error drops below 0.8×10^{-6} ft.

To obtain a satisfactory steady-state disturbance accommodation, it is necessary that the controller and observer Riccati equation possess a quasiconstant solution according to Eqs. (65) and (66). To demonstrate this, we consider an extreme case. With reference to Eq. (62), we use a second-order model where $\varphi(t) = 0.25t^2$ $(\text{rad/s})^2$, $t \in [0,2.5)$, $\omega_1 = 0.2$ rad/s, $\|\bar{K}_{22}\| \cong 1$, so that $\varphi(t)\|\bar{K}_{22}\| > \omega_1$ for $t > 0.4$ s. Figures 4 and 5 show plots of $S_{11}(t)/\bar{S}_{011}$ vs t for various values of α, where $S_{11}(t)$ is the entry of $S(t)$, Eq. (19), exhibiting the largest deviation relative to the corresponding entry, $\bar{S}_{011}$, of the algebraic Riccati matrix $\bar{S}_0$. It is obvious from Fig. 4 that $\varepsilon_1(\alpha)$, Eq. (63), decreases for increasing α. Finally, Fig. 5 shows that, by choosing $S(t_f) = \bar{S}_0$, the transient Riccati solution approaches the algebraic solution for $t \in \tau$, as indicated by Eq. (65). From Fig. 5, we see that the maximum deviation from the algebraic solution is smaller than 5% for $\alpha = 4$ and smaller than 0.5% for $\alpha = 12$.

The design policy of the control law is demonstrated next. The gain matrices $K_C(t)$, $K_E(t)$, and K_D are calculated according to Eqs. (18b), (36), and (60), respectively, the convergence factor β is calculated according to $\beta = 3\alpha$, and the weighting matrices Q and R in Eq. (14) and Q_E in Eq. (37a) are taken as unit matrices. The matrix L_y in Eq. (44) consists of slopes at the locations $y_j = jL$ ($j = 1.0$, 0.85, 0.75, 0.65, 0.55, 0.35, 0.25). Figure 6 shows the closed-loop response of the modeled system in case of an eighth-order controlled model for $p = n_C/2 = 4$, where the locations of the actuators are $y_i = iL/4$ ($i = 1,2,3,4$). Figure 7 shows a similar plot for a single actuator located at $y = L/2$. The optimal values of α yielding the minimal supremum time constant are 5 and 2, respectively. As $t \to t_h$, the displacement errors drop below 0.008 and 0.017 ft, respectively. As expected, the error in steady state is larger and the optimal convergence factors are smaller in the case of a single actuator.

Finally, Fig. 8 represents the total angular displacement at the tip of the beam for an 180-deg maneuver and correspond-

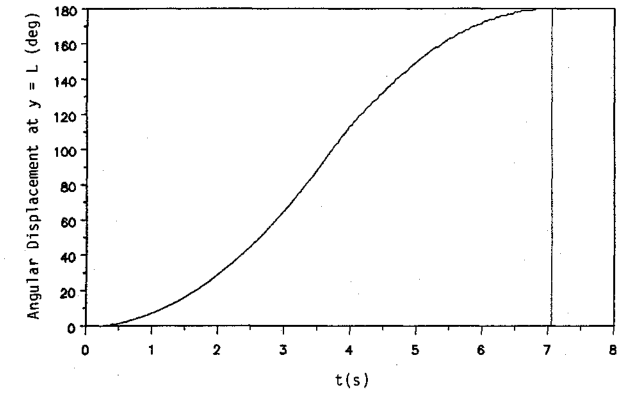

Fig. 8 Angular displacement at the tip ($p = 3$, $\alpha = 3$); maneuver angle = 180 deg.

ing to the case of a sixth-order controlled model with $p = 3$ and $\alpha = 3$. Deviations from the ideal on-off maneuver due to flexibility are well below the resolution of the graph.

IX. Conclusions

This paper presents a perturbation approach to the problem of maneuvering of a flexible spacecraft. The zero-order problem consists of the rigid-body maneuvering of the spacecraft, and the first-order problem consists of control of the elastic motion and of perturbations in the rigid-body motions, where "zero" and "first" refer to order of magnitude in the context of a perturbation scheme. The rigid-body maneuvering is carried out in minimum time according to a bang-bang control law, which is well documented in the technical literature, so that the emphasis in this paper is on the first-order problem. The main conclusions are the following:

1) A near minimum-time maneuvering and control is achieved, where the zero-order model is maneuvered according to a minimum-time policy and the first-order model is controlled by means of a reduced-order compensator during the finite time of the maneuver.

2) The first-order model, which is linear, time-dependent, and of high order, is controlled by a reduced-order compensator yielding a) an optimally controlled reduced-order model according to a finite-time quadratic performance measure, b) minimization of a measure of the disturbance response of the full first-order model toward the end of finite-time intervals, and c) minimization of the supremum time constant of the elastic model.

3) The characteristics just mentioned are achieved toward the end of the finite-time intervals through the inclusion of a convergence factor in the Riccati equation both for the controller and for the observer. As a result, as the convergence factors increase, the steady-state solutions of the controller and of the observer Riccati equation converge to the corresponding algebraic solutions.

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Appendix: Finite-Time Stability

The ordinary stability definitions relate to systems operating over an infinite time interval. For finite-time stability, we must consider different definitions. Under consideration is the linear system

$$\dot{z} = A(t)z, \qquad z(t_0) = z_0 \tag{A1}$$

defined over the time interval $\nu = [t_0, t_f)$, where $A(t)$ is an $n \times n$ matrix continuous on $R^n \times \nu$ and bounded for all $t \in \nu$. Then, we denote by $\bar{z}(t^*, t, z)$ a trajectory of Eq. (A1) evaluated at time t^*, which takes on the value z at time t, and by $\| \ \|$ a norm on $R^n \times \nu$.

Definition 1

System (A1) is exponentially contractively stable (ESC) with respect to $[\delta, \psi, \alpha, \mu(t), \nu, \| \ \|]$, where α is a positive constant, $\mu(t)$ is a continuous positive function and $\psi \geq \delta$, if every trajectory $\bar{z}(t^*, t_0, z_0)$ for which $\|z_0\| < \delta$ is such that

$$\|z(t, t_0, z_0)\| < \psi \exp\left[-\int_{t_0}^{t} \mu(s)\, \mathrm{d}s\right] \exp[-\alpha(t - t_0)], \quad t \in \tau \tag{A2}$$

Definition 2[26]

System (A1) is quasicontractively stable (QCS) with respect to $[\delta, \rho, \nu, \| \ \|]$ if for every trajectory $\bar{z}(t^*, t_0, z_0)$, where $\|z_0\| < \delta$, there exists a $T_1 \in (t_0, t_f)$ such that

$$\|z(t^*, t_0, z_0)\| < \rho, \quad \rho < \delta, \qquad t^* \in (T_1, t_f) \tag{A3}$$

Definition 3

System (A1) is strictly finite-time stabilizable (SFTS) with respect to $[\delta, \varepsilon, \nu, \| \ \|]$ if for any positive ε there exists a $T_1 \in (t_0, t_f)$ and every trajectory $z(t^*, t_0, z_0)$ for which $\|z_0\| < \delta$ has the property

$$\|\bar{z}(t^*, t_0, z_0)\| < \varepsilon \text{ for all } t^* \in (T_1, t_f) \tag{A4}$$

At this point, we wish to formalize sufficiency conditions for ECS and SFTS systems. With reference to Eq. (A1), we define

$$z^* = e^{\gamma t} z, \qquad \gamma \geq 0 \tag{A5}$$

so that Eq. (A1) yield

$$\dot{z}^* = [A(t) + \gamma I]z^*, \qquad z^*(t_0) = z_0^* \tag{A6}$$

Next, we introduce a Lyapunov function

$$V(z^*, t) = z^{*T} B(t) z^* \tag{A7}$$

where $B(t)$ is a symmetric and differentiable matrix such that

$$0 \leq c_1 \leq \|B(t)\| \leq c_2 \leq \infty, \qquad t \in \nu \tag{A8}$$

Differentiating Eq. (A7) with respect to time and evaluating along a trajectory of Eq. (A6), we obtain

$$\dot{V}(z^*, t) = -z^{*T} C(t) z^* \tag{A9}$$

where $C(t)$ is a symmetric differentiable matrix. Then, it can be shown[17] that

$$\|z(t)\| < \|z_0\| r_B^{1/2}(t) \exp\left[-\frac{1}{2}\int_{t_0}^{t} \mu(s)\, \mathrm{d}s\right] \exp[-\gamma(t - t_0)] \tag{A10}$$

where

$$r_B(t) = \frac{\max_i \lambda_i[B(t_0)]}{\min_i \lambda_i[B(t)]}, \qquad \mu(t) = \frac{\min_i \lambda_i[C(t)]}{\max_i \lambda_i[B(t)]} \tag{A11}$$

These developments, in conjunction with Definitions 1 and 3, permit us to state the following sufficiency conditions:

1) If $C(t) > 0$ for all $t \in \nu$, Eq. (A1) represent an ECS system.

2) If $C(t) > 0$ for all $t \in \nu$, and in addition

$$\delta r_B^{1/2}(t) \exp[-\gamma(t - t_0)] < \varepsilon \text{ for all } t \in (T_1, t_f), \quad t_0 < T_1 < t_f \tag{A12}$$

then Eq. (A1) represents an SFTS system.

Finally, we can rewrite inequality (A10) in the form

$$\|z(t)\| < \|z_0\| \bar{r}_B^{1/2} \exp\left[-\frac{1}{2}\int_{t_0}^{t} \mu(s)\, \mathrm{d}s\right] \exp[-\gamma(t - t_0)] \tag{A13}$$

where $\bar{r}_B = \max_{t \in \tau}(r_B(t))$. It is clear from inequality (A13) that if $\mu(t) > 0$ for all $t \in \tau$, then $\|z(t)\|$ converges to zero not slower than $e^{-\gamma t}$. Therefore, $1/\gamma$ may be interpreted as an upper bound on the time constant of system (A1). We denote by π the least upper bound of $1/\gamma$ and refer to π as the supremum time constant (STC). It is clear that the STC corresponds to the value of γ for which $\mu(\gamma, t) = 0$ for some $t \in \tau$. Hence,

$$\pi = 1/\bar{\gamma} \tag{A14}$$

where $\bar{\gamma}$ is the maximum value of γ for which

$$\min_{t \in \tau} |\lambda_m[C(\bar{\gamma}, t)]| = 0 \tag{A15}$$

and where λ_m denotes the minimum eigenvalue of the argument matrix.

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